LAMINAR CONVECTION OVER A LINEAR HEAT SOURCE

(LAMINARNAIA KONVEKTSIIA NAD LINEINYM ISTOCHNIKOM TEPLA)

PMM Vol.22, No.4, 1958, pp.573-576

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(Received 20 November 1957)

An asymptotic solution to the problem of laminar convection of heat over a linear source of heat is given. It would appear that Zeldovich [1] was the first to deal with this problem. In 1937 he obtained formulas for the longitudinal velocity component and liquid temperature from dimensional considerations, without even solving the convection equations.

1. Basic Problem and its Equations. We deal with the problem of steady thermal convection of a liquid over an infinitely long straight horizontal heated wire or thread.

The heat emitted by the thread causes nonuniform heating of the liquid surrounding it, which leads to the convective movement of the liquid. The liquid flows in the form of a rising laminar jet which expands with height. When a certain height is attained the laminar flow breaks down and is replaced by mixing and turbulence.

In studying the laminar flow we may use existing knowledge of the effect of large velocity and temperature gradients transverse to the jet, a characteristic feature of the boundary layer close to a wall.

The equations of free convection can therefore be simplified considerably [2] by making use of the approximations in boundary layer theory. They may be written

$$(\mathbf{v}\nabla)\,\mathbf{v} = -\nabla\frac{p'}{\rho} + \nu\Delta\mathbf{v} - \mathbf{g}\beta T', \qquad \mathbf{v}\nabla T' = \chi\Delta T', \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

Here $\mathbf{v}$ is the velocity, T and p' are the deviations in temperature and pressure from their equilibrium values, ρ is the liquid density, g, gravity, β is the coefficient of thermal expansion, ν and χ are the coefficients of viscosity and conductivity of the liquid. The differential equations of motion of the jet can be written;

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta T + \nu\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \chi\frac{\partial^2 T}{\partial y^2} \tag{1.2}$$

Here, u and v are the velocity components of the fluid along the axes of x and y (x is vertically upwards in the plane of symmetry of the jet, y is transverse to the jet, and z is in the direction of the source). The term containing the pressure is omitted because the pressure only changes in the direction of the jet and equals the hydrostatic pressure $p(x)$, and therefore $p' = 0$.

We solve equations (1.2) under the following boundary conditions:

$$v = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial T}{\partial y} = 0 \quad \text{for } y = 0 \tag{1.3}$$

$$u = 0, \quad T = 0 \quad \text{for } y = \infty \tag{1.4}$$

The last two conditions (1.3) express the fact that the plane xz is a plane of flow symmetry.

We now introduce a quantity representing the strength of the source, for example, the quantity of heat Q emitted per unit time by unit length of thread. Then we express the constant heat flux across any horizontal plane in the form (molecular heat transfer is neglected);

$$Q = 2c\rho \int_0^\infty uT\,dy = \text{const} \tag{1.5}$$

Here c is the specific heat of the liquid. We can introduce the stream function ψ by means of the equation of continuity,

$$u = \partial\psi / \partial y, \qquad v = -\partial\psi / \partial x \tag{1.6}$$

There is no characteristic length in this problem. It seems natural therefore to assume that the profiles of velocity $u(x, y)$ and temperature $T(x, y)$ are geometrically similar, i.e. they can be made to coincide in any sections of the flow $x =$ constant, if suitable scales are chosen for u, T and y.

Bearing this in mind, and assuming that the terms in the first of equations (1.2) are of the same order of magnitude, and also assuming that the heat emitted Q is independent of x, we seek solutions for the stream function and temperature in the form

$$\psi = 5\left[\frac{Qg\beta\nu^2}{50c\rho}\right]^{1/5} x^{3/5} f(\eta) \tag{1.7}$$

$$\left(\eta = \left[\frac{Og\beta}{50c\rho\nu^3}\right]^{1/5} x^{-2/5} y\right)$$

$$T = \frac{5\nu^2}{g\beta}\left[\frac{Qg\beta}{50c\rho\nu^3}\right]^{4/5} x^{-3/5} \varphi(\eta) \tag{1.8}$$

Here η is a nondimensional coordinate and f and ϕ are nondimensional functions.

The f:ctors are chosen to simplify the equations in the transformations which follow below.

Substituting (1.6) and (1.8) in (1.2), we arrive at a system of ordinary differential equations for the functions f and ϕ,

$$f''' + 3ff'' = f'^2 - \varphi, \qquad 36\,(f\varphi)' = -\,\varphi'' \qquad \left(\sigma = \frac{\nu}{\chi}\right) \tag{1.9}$$

where σ is the Prandtl number, and a prime denotes differentiation with respect to η.

Boundary conditions (1.3) and (1.4) in the new variables are

$$f = f'' = 0, \qquad \varphi' = 0 \qquad \text{for } \eta = 0 \tag{1.10}$$

$$f' = 0, \qquad \varphi = 0 \qquad \text{for } \eta = \infty \tag{1.11}$$

2. Integration of the Equations. We begin by integrating the second equation (1.9) twice. Using boundary conditions (1.10) the integrated equation can be written

$$\varphi = \varphi_0 \exp\left(-\,3\sigma \int_0^\eta f d\eta\right) \tag{2.1}$$

where $\phi_0 = \phi(0)$, a magnitude to be evaluated later.

If we substitute (2.1) in the first equation (1.9) we obtain

$$f''' + 3ff'' = f'^2 - \varphi_0 \exp\left(-\,3\sigma \int_0^\eta f d\eta\right) \tag{2.2}$$

We seek a solution of this equation in the form of a power series;

$$f = \sum_{n=0}^{\infty} \frac{a_n}{n!}\,\eta^n \tag{2.3}$$

Using the boundary conditions (1.10) for $\eta = 0$ we immediately find that $a_0 = a_2 = 0$. If we substitute (2.3) in (2.2) and equate coefficients of like powers of η in both sides of the equations, we can express all the coefficients of the series (2.3) in terms of $a_1 = a$ and ϕ_0;

$$a_3 = a^2 - \varphi_0, \quad a_4 = 0, \quad a_5 = 3\sigma\varphi_0 a - 4a\,(a^2 - \varphi_0), \quad a_6 = 0 \tag{2.4}$$
$$a_7 = 34a^4 - a^2\varphi_0\,(28 + 27\sigma + 27\sigma^2) - 3\varphi_0^2\,(2 + \sigma), \ldots$$

The fact that all the coefficients of even powers are zero can also be explained as follows. The function $f'(\eta)$, which is proportional to the velocity u, must be an even function of variable η, and therefore f is odd, so that coefficients of even powers of η in (2.3) must be zero. Series (2.3) can be written

$$f = \sum_{n=0}^{\infty} \frac{a_{2n+1}}{(2n+1)!}\,\eta^{2n+1} \tag{2.5}$$

Treating (2.2) as a linear nonhomogeneous equation in f'' and its derivative, and noting that $f''(0) = 0$, we obtain the solution

$$f'' = e^{-3F(\eta)}\,\Phi(\eta) \tag{2.6}$$

where

$$F(\eta) \equiv \int_0^\eta f\,d\eta, \qquad \Phi(\eta) \equiv \int_0^\eta (f'^2 - \varphi_0 e^{-3\sigma F(\eta)})\, e^{3F(\eta)} d\eta \tag{2.7}$$

To satisfy boundary condition (1.11) we integrate (2.6) from 0 to ∞, and obtain;

$$f'(\infty) - f'\,0) = \int_0^\infty e^{-3F(\eta)}\,\Phi(\eta)\,d\eta$$

or

$$\int_0^\infty e^{-3F(\eta)}\,\Phi(\eta)\,d\eta = -a \tag{2.8}$$

We get the second equation with the same unknowns a and ϕ_0 from condition (1.5), which, in terms of variables f, ϕ, η takes the form:

$$\varphi_0 \int_0^\infty e^{-3\sigma F(\eta)} f'\,d\eta = 1 \tag{2.9}$$

The integrals (2.8) and (2.9) cannot be expressed in terms of elementary functions but can, in certain cases, be expanded asymptotically using an iterative method.

In this case $F(\eta)$ is a positive function which tends monotonically to infinity together with η, and possesses a stationary point $\eta = 0$. The functions $f'(\eta)$ and $\Phi(\eta)$ are smooth. It is possible to use this iterative method, therefore, for the approximate integration of (2.8) and (2.9). The method is similar to that used by Watson [3] and by Meksyn [4] for integrating the boundary layer equations.

The method of working out integrals (2.8) and (2.9) involves a new variable of integration τ, where

$$F(\eta) = \tau \tag{2.10}$$

or, replacing $F(\eta)$ by the power series, given by (2.7)

$$\sum_{n=0}^{\infty} \frac{a_{2n+1}}{(2n+2)!}\,\eta^{2n+2} = \tau \tag{2.11}$$

We now invert the series (2.11). Using known formulas obtained from Dwight's handbook [5] for the coefficients, we first of all find the expression

$$\eta^2 = \left(\frac{a}{2}\right)^{-1}\tau - \frac{a_3}{24}\left(\frac{a}{2}\right)^{-3}\tau^2 + \frac{1}{1440}(5a_3^2 - aa_5)\left(\frac{a}{2}\right)^{-5}\tau^3 - \\ - \frac{1}{483840}(3a^2a_7 - 70aa_3a_5 + 175a_3^3)\left(\frac{a}{2}\right)^{-7}\tau^4 + \ldots] \tag{2.12}$$

and then find η,

$$\eta = \left(\frac{a}{2}\right)^{-1/2}\tau^{1/2}\left[1 - \frac{a_3}{48}\left(\frac{a}{2}\right)^{-2}\tau + \frac{1}{23040}(35a_3^2 - 8aa_5)\left(\frac{a}{2}\right)^{-4}\tau^2 + \ldots\right] \tag{2.13}$$

Substituting (2.12) in the series $f' = \sum_{n=0}^{\infty} \frac{a_{2n+1}}{(2n)!}\eta^{2n}$, we obtain

$$f' = a + \frac{a_3}{2}\left(\frac{a}{2}\right)^{-1}\tau + \frac{1}{48}(aa_5 - a_3^2)\left(\frac{a}{2}\right)^{-3}\tau^2 + \\ + \frac{1}{2880}(a^2a_7 - 6aa_3a_5 + 5a_3^3)\left(\frac{a}{2}\right)^{-5}\tau^3 + \ldots \tag{2.14}$$

It only remains to transform the function $\Phi(\eta)$ to the new variable. We use equation (2.6) to do this, rewriting it in the form

$$\frac{df'}{d\tau} = e^{-3\tau}\,\Phi(\eta)\frac{d\eta}{d\tau} \tag{2.15}$$

Let us put

$$\Phi(\eta)\frac{d\eta}{d\tau} = \sum_{m=1}^{\infty} b_m \tau^{1/2(m-1)} \tag{2.16}$$

Substituting (2.14) and (2.16) in equation (2.15) and equating coefficients of equal powers of τ on both sides, we find;

$$b_1 = \frac{1}{2}\left(\frac{a}{2}\right)^{-1}(a^2 - \varphi_0), \qquad b_2 = 0, \qquad b_3 = \frac{1}{24}\left(\frac{a}{2}\right)^{-3}[4a^4 + 3a^2\varphi_0(\sigma - 1) - \varphi_0^2] \quad b_4 = 0 \\ b_5 = \frac{1}{260}\left(\frac{a}{2}\right)^{-5}(48a^6 - 46a^4\varphi_0 + 45\sigma a^4\varphi_0 - 27\sigma^2a^4\varphi_0 + 3a^2\varphi_0^2 + \\ + 15\sigma a^2\varphi_0^2 - 5\varphi_0^3) \qquad b_6 = 0, \ldots \tag{2.17}$$

Thus, in order to integrate (2.8) we have

$$\int_0^{\infty} e^{-3F(\eta)}\Phi(\eta)\,d\eta = \int_0^{\infty} e^{-3\tau}\sum_{m=1}^{\infty} b_m \tau^{\frac{1}{2}(m+1)-1}\,d\tau = \sum_{m=1}^{\infty} b_m 3^{-\frac{m+1}{2}}\Gamma\left(\frac{m+1}{2}\right) \tag{2.18}$$

where Γ is the Gamma function. Substituting (2.18) in (2.8) we get

$$\sum_{m=1}^{\infty} b_m 3^{-\frac{m+1}{2}}\Gamma\left(\frac{m+1}{2}\right) = -a \tag{2.19}$$

Using (2.13) and (2.14) we find the integral (2.9) in a similar manner; with the result

$$\varphi_0 \sum_{m=0}^{\infty} c_m (3\sigma)^{-\frac{m+1}{2}}\Gamma\left(\frac{m+1}{2}\right) = 1 \tag{2.20}$$

where

$$c_0 = \left(\frac{a}{2}\right)^{1/2}, \qquad c_1 = 0, \qquad c_2 = \frac{3a_3}{16}\left(\frac{a}{2}\right)^{-3/2}, \qquad c_3 = 0$$

$$c_4 = \frac{1}{4608}\left(\frac{a}{2}\right)^{-7/2}[-245a^4 + 10a^2\varphi_0(33 + 12\sigma) - 85\varphi_0^2], \qquad c_5 = 0, \ldots \qquad (2.21)$$

In our case it appears that the series (2.19) and (2.20) converge rapidly, and therefore a sufficiently good approximation for a and ϕ_0 can be obtained by using three or four terms of the series.

We therefore arrive at the solution which formally satisfies all the boundary conditions in the form of the series

$$f = a\eta + \frac{1}{6}(a^2 - \varphi_0)\eta^3 + \frac{1}{120}[3\sigma a\varphi_0 - 4a(a^2 - \varphi_0)]\eta^5 + \\ + \frac{1}{5040}[34a^4 - a^2\varphi_0(28 + 27\sigma + 27\sigma^2) - 3\varphi_0^2(2 + \sigma)]\eta^7 + \ldots \qquad (2.22)$$

$$\varphi = \varphi_0\{1 - \frac{3}{2}\sigma a\eta^2 + \frac{1}{8}\sigma[(9\sigma - 1)a^2 + \varphi_0]\eta^4 - \frac{1}{240}a\sigma[45\sigma a^2(3\sigma - 1) + \\ + 4\varphi_0(12\sigma + 1) - 4a^2]\eta^6 + \ldots\} \qquad (2.23)$$

The solution to our problem has only limited application. In the first place it cannot be applied in the immediate vicinity of the source. Any real source has finite thickness and this solution only begins to be valid at a distance from the source where its dimensions do not significantly affect the flow of the fluid. Secondly, the flow becomes turbulent at some given height.

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Translated by V.H.B.